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Stratified order one differential equations in positive characteristic

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ABSTRACT

Stratification for nonlinear differential equations in positive characteristic is introduced. Testing this notion for first order equations is discussed, and related to the Cartier operator on curves. A variant of the Grothendieck–Katz conjecture is formulated, and proved in a special case.

0. Introduction

Let F be a field of characteristic $p > 0$. A *higher derivation* on F is a set of additive maps $\{\partial^{(n)} : F \to F \mid n \geq 0\}$ satisfying the formulas

$$\partial^{(n)}(fg) = \sum_{\substack{a,b\geq 0 \\ a+b=n}} \partial^{(a)}(f)\partial^{(b)}(g)$$

and

$$\partial^{(0)} = id; \qquad \partial^{(n)}\partial^{(m)} = \binom{m+n}{n}\partial^{(n+m)}.$$

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The *standard higher derivation* $\{\partial_z^{(n)} \mid n \geq 0\}$ on the field $C(z)$, where C is an algebraically closed field of characteristic $p > 0$, is given and determined by the formulas $\partial^{(n)} z^m = \binom{m}{n} z^{m-n}$ for all $n, m \geq 0$. These formulas imitate the maps $\frac{1}{n!}(\frac{d}{dz})^n$. The standard higher derivation on $C(z)$ extends uniquely to a higher derivation on any finite separable extension K of $C(z)$.

The algebra of the differential operators $\mathcal{D} = \mathcal{D}_K$ on the field K (see Grothendieck, 1967, § 16, particularly 16.10; Gieseker, 1975 for the concept and terminology) is a skew K-algebra and has K-basis $\{\partial_z^{(n)} \mid n \geq 0\}$. For the purely transcendental extension $C(z_1, \ldots, z_d)$ of C, one considers operators $\partial_{z_1}^{(n_1)} \cdots \partial_{z_d}^{(n_d)}$ with similar formulas

$$\partial_{z_1}^{(n_1)} \cdots \partial_{z_d}^{(n_d)} \left(z_1^{m_1} \cdots z_d^{m_d}\right) = \binom{m_1}{n_1} \cdots \binom{m_d}{n_d} z_1^{m_1 - n_1} \cdots z_d^{m_d - n_d}.$$

This is the standard higher derivation of $C(z_1, \ldots, z_d)$ and extends uniquely to any finite separable extension K of $C(z_1, \ldots, z_d)$. As before the algebra of differential operators of K has K-basis

$$\left\{\partial_{z_1}^{(n_1)} \cdots \partial_{z_d}^{(n_d)} \,\middle|\, n_1, \ldots, n_d \geq 0\right\}.$$

A *stratified differential module* M over K is a left $\mathcal{D}$-module which has finite dimension over K. This concept coincides with that of *iterative differential module* (ID-module) over K (see Matzat and van der Put, 2003a, 2003b). This notion is seen as the correct analogue in positive characteristic of complex differential modules.

The *naive translation* of a differential module in positive characteristic is a K-vector space M of finite dimension over K equipped with an additive map $\partial : M \to M$ satisfying $\partial(fm) = f\partial(m) + \frac{df}{dz} m$ (where K a finite separable extension of $C(z)$).

This naive approach does not have good properties and is not very interesting. In fact, M as a naive differential module is determined by its p-curvature ∂^p (compare van der Put, 1995, Prop. 2.1). The action of ∂ on M extends to a left action of $\mathcal{D}$, if and only if $\partial^p = 0$. Indeed, this condition is necessary since $(\partial^{(1)})^p = 0$. It is sufficient since, by Cartier's lemma (see van der Put, 2001, Lemma 1.2), $\partial^p = 0$ implies that M is a trivial differential module and has therefore a left $\mathcal{D}$-action, for instance the (non-interesting) trivial one.

In this paper we introduce the notion of stratification for nonlinear differential equations. We restrict ourselves mainly to nonlinear order one differential equations over a finite separable extension K of $C(z)$. Such an equation has the form $f(y', y) = 0$ where $f \in K[S, T]$ is an absolutely irreducible polynomial such that the image d of $\frac{df}{dS}$ in $K[S, T]/(f)$ is nonzero. The differential algebra $A := K[y', y, \frac{1}{d}] = K[S, T, (\frac{df}{dS})^{-1}]/(f)$ is given by the derivation D with $D(z) = 1$ and $D(y) = y'$.

A *separable algebraic solution* of f is a K-linear differential homomorphism $\phi : A \to \overline{K}^{sep}$ with $\phi(1) = 1$. Then $\phi(y)$ is the actual solution.

The derivation D on A extends uniquely to its field of fractions F. We note that F is the function field of a curve X over K. By *the genus of* f we will mean the genus of X.

Two differential equations over K will be called *strictly equivalent* if the corresponding differential fields become isomorphic after a finite separable extension of K (compare Ngo et al., 2013).

The equation $f(y', y) = 0$ is called *stratified* (or admits a stratification) if the field F (or equivalently A) has a left $\mathcal{D}_K$-action such that the action of $\partial^{(1)}$ coincides with D. Alternatively, A should be an Artinian simple module algebra over the Hopf algebra D_K in the sense of Heiderich, 2013, Example 3.2(5); Example 3.4(6); Example 3.6. See also Amano and Masuoka, 2005; Heiderich, 2014.

We will show that $f(y', y) = 0$ is stratified if and only if $D^p = 0$. This leads to criteria for stratification. Further we try to classify the stratified order one differential equations. A useful observation for this is: if f_1 and f_2 are strictly equivalent and f_1 is stratified, then so is f_2.

The order one equation $f(y', y) = 0$ is called *autonomous* if $f \in C[S, T]$. The equation $f(y', y) = 0$ is called *semi-autonomous* if there exist a finite separable extension $\tilde{K}$ of K and a curve X_0 over C such that curve X over K defined by f satisfies $\tilde{K} \times_K X \cong \tilde{K} \times_C X_0$.

The aim of this paper, which is a sequel to Ngo et al. (2013), is to produce and classify stratified order one differential equations in positive characteristic. In the last section, a conjecture in the spirit of the Grothendieck–Katz conjecture is formulated. A proof is given for autonomous equations.

1. Criteria for stratification

Theorem 1.1. *Let K be a finite separable extension of $C(z)$ and let $f \in K[S,T]$ be absolutely irreducible. Suppose that $\frac{df}{dS}$ is not zero modulo (f). Let D denote the derivation on $F := K(y', y)$, the field of fractions of $K[S,T]/(f)$, such that $D(z)=1$, $D(y)=y'$. The following are equivalent:*

(a) *The equation $f(y', y) = 0$ is stratified.*
(b) $D^p = 0$.
(c) $\ker(D, F)$ *contains an element $h \notin F^p$.*
(d) *$f(y', y) = 0$ has infinitely many separable algebraic solutions.*

Proof. (a) $\Rightarrow$ (b) is obvious because $D^p = (p!)\partial^{(p)}$.

(b) $\Rightarrow$ (c). Suppose that $D^p = 0$. The field F is a finite separable extension of the purely transcendental field $C(z, y)$ and therefore $[F : F^p] = p^2$. The derivation D is a F^p-linear map on F. Since $D^p = 0$ and $[F : F^p] = p^2$, the operator D has more than one Jordan block. Thus $\tilde{F} = \ker(D, F)$ lies properly in between F^p and F.

(c) $\Rightarrow$ (a). Let $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial y}$ denote the obvious partial derivations on $C(z, y)$. Their unique extensions to the finite separable extension F of $C(z, y)$ are denoted by the same symbols. The derivation D has the form $D = \frac{\partial}{\partial z} + y' \frac{\partial}{\partial y}$. Let $D(h) = 0$ hold for some element $h \notin F^p$. Then $\frac{\partial h}{\partial z} + y' \cdot \frac{\partial h}{\partial y} = 0$. The assumption $\frac{\partial h}{\partial y} = 0$ implies $\frac{\partial h}{\partial z} = 0$ and $h \in F^p$. Thus $\frac{\partial h}{\partial y} \neq 0$ and $\frac{\partial h}{\partial z} \neq 0$. The element h is not algebraic over K, since otherwise h would be separable algebraic over K and $\frac{\partial h}{\partial y} = 0$. Now $K(h) \subset F$ is a finite algebraic extension, since both fields have transcendence degree two over C and F is finitely generated. If the field $F^p(z, h)$ is a proper subfield of F, then there is a derivation $E := \alpha \frac{\partial}{\partial z} + \beta \frac{\partial}{\partial y} \neq 0$ which is zero on $F^p(z, h)$. Applying E to z and h yields $\alpha = \beta = 0$. Thus $F^p(z, h) = F$ and $K(h) \subset F$ is a finite separable extension. On the field $K(h)$ we define the action of $\mathcal{D}_K = K[\partial^{(n)} \mid n \geq 0]$ by $\partial^{(0)}$ is the identity, $\partial^{(n)}$ is on K the given higher derivation and $\partial^{(n)} h = 0$ for all $n \geq 1$. This action extends in a unique way to an action of $\mathcal{D}_K$ on F. By construction $\partial^{(1)}$ coincides with D on $K(h)$ and therefore also on F.

(c) $\Rightarrow$ (d). We continue the reasoning of (c) $\Rightarrow$ (a). After multiplying h by a nonzero element in F^p, we may suppose that $h \in A := K[y', y, \frac{1}{d}]$. The field extension $K(h) \subset F$ is finite separable. After localization at an element $k \in K[h]$, $k \neq 0$ we have that $K[h, \frac{1}{k}] \subset A[\frac{1}{k}]$ is a finite separable integral extension. For almost all $c \in C$ there is a K-algebra homomorphism $\psi : K[h, \frac{1}{k}] \to K$, given by $\psi(h) = c$. This extends to a K-algebra homomorphism $\phi : A[\frac{1}{k}] \to \overline{K}^{sep}$. By definition $\psi \circ D = \frac{d}{dz} \circ \psi$ (where $\frac{d}{dz}$ denotes the derivation on $\overline{K}^{sep}$ with $\frac{d}{dz} z = 1$). The same holds for ϕ, i.e., $\phi \circ D = \frac{d}{dz} \circ \phi$ and thus ϕ is a separable algebraic solution. By varying $c \in C$ one obtains infinitely many separable algebraic solutions of $f(y', y) = 0$.

(d) $\Rightarrow$ (b). Suppose that $f(y', y) = 0$ has infinitely many separable algebraic solutions and suppose that $D^p \neq 0$. There exists $a \in A = K[y', y, \frac{1}{d}]$ with $b := D^p(a) \neq 0$. Let $\phi : A \to \overline{K}^{sep}$ be a separable algebraic solution. Then $\phi \circ D = \frac{d}{dz} \circ \phi$ and then also $\phi \circ D^p = (\frac{d}{dz})^p \circ \phi$. Since $(\frac{d}{dz})^p = 0$ one has $\phi(D^p(a)) = 0$. Since the K-algebra A/bA has Krull dimension zero, there are at most finitely many K-algebra homomorphisms $A/bA \to \overline{K}^{sep}$, contradicting the assumption (d). □

Note that property (c) is usually called "existence of a rational first integral".

Now we consider a criterion for stratification of autonomous equations.

Proposition 1.2. *Let F be a field of characteristic $p > 0$ such that $[F : F^p] = p$ and let $D \neq 0$ be a derivation on F. Write $F = F^p(x)$ and $u = D(x)$. The following are equivalent:*

(a) *D extends to a higher derivation on F.*
(b) $D^p = 0$.

(c) *There exists $t \in F$ with $D(t) = 1$.*
(d) $u^{-1} = a_0 + a_1 x + \cdots + a_{p-2}x^{p-2}$ *with all* $a_* \in F^p$.
(e) $(\frac{d}{dx})^{p-1}(u^{p-1}) = 0$.

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c). Suppose $D^p = 0$. The operator D is a F^p-linear on F and $\ker D \supset F^p$ is a field. Hence $\ker D = F^p$ and D has precisely one Jordan block. Hence $1 \in F^p$ lies in the image of D.

(c) $\Rightarrow$ (a). Suppose that $D(t) = 1$ holds for some t. Then $F = F^p(t)$ and there is a higher derivation $\{\partial_t^{(n)} \mid n \geq 0\}$, given by the formulas $\partial_t^{(n)}(t^m) = \binom{m}{n} t^{m-n}$. Now $D = \frac{d}{dt}$ coincides with $\partial_t^{(1)}$.

(c) $\Leftrightarrow$ (d). $D = u \cdot \frac{d}{dx}$. The equation $D(t) = 1$ translates into $\frac{d}{dx} t = u^{-1}$ and there is a solution if and only if u^{-1} has the prescribed form.

Further (d) is equivalent to $(\frac{d}{dx})^{p-1}(u^{-1}) = 0$ and after multiplying with u^p one obtains (d) $\Leftrightarrow$ (e). □

Corollary 1.3 *(Autonomous equations). Let the autonomous first order equation $f(y', y) = 0$ determine the field $C(y', y)$ provided with the derivation D given by $D(y) = y'$. The following are equivalent:*

(a) *The equation is stratified.*
(b) *There exists $t \in C(y', y)$ with $D(t) = 1$.*
(c) *The equation has a separable algebraic solution.*

Proof. (a) $\Leftrightarrow$ (b) holds according to Proposition 1.2. Suppose $D(t) = 1$ for some $t \in C(y', y)$. Then $C(t) \subset C(y', y)$ is a separable extension and the embedding $C(t) \to C(z)$, by $t \mapsto z + c$ with $c \in C$, extends to an embedding of $C(y', y)$ into $\overline{C(z)}^{sep}$. Thus the equation has a family of separable algebraic solutions. This shows (b) $\Rightarrow$ (c).

Suppose that (c) holds. Then there is an embedding $C(y', y) \subset \overline{C(z)}^{sep} = \overline{C(y', y)}^{sep}$ and the equation $D(t) = 1$ over the field $C(y', y)$ has a separable algebraic solution, say z. Let G denote the Galois group of $\overline{C(y', y)}^{sep}/C(y', y)$. For any $g \in G$ one has $g(z) = z + c(g)^p$ for a unique element $c(g) \in \overline{C(y', y)}^{sep}$. Now $g \mapsto c(g)$ is a 1-cocyle and trivial since $H^1(G, \overline{C(y', y)}^{sep}) = 0$. Write $c(g) = g(h) - h$ for some $h \in \overline{C(y', y)}^{sep}$. Then $t := z - h^p$ satisfies $D(t) = 1$ and $t \in C(y', y)$ since $g(t) = t$ for all $g \in G$. Hence (c) $\Rightarrow$ (b). □

2. Testing the existence of a stratification for autonomous equations

An autonomous equation translates into the function field $C(X)$ of a curve X over C, provided with a meromorphic vector field $D \neq 0$ (i.e., a derivation of the field $C(X)$). The equation is stratified if and only if $D^p = 0$. There are many autonomous stratified equations and we are interested in equations having a geometric meaning. These equations will be called “special”. If the genus of X is zero, then the holomorphic vector fields are special.

Example 2.1. *The autonomous Riccati equation* $y' + a_2 y^2 + a_1 y + a_0 = 0$.

By 1.2 and 1.3, this equation is stratified if and only if the coefficient of y^{p-1} in the expression $(a_2 y^2 + a_1 y + a_0)^{p-1}$ is zero. A computation shows that this is equivalent to $(a_2 y^2 + a_1 y + a_0)$ is a square.

The general case is as follows.

Corollary 2.2. *The autonomous equation $y' r(y) = 1$ with $r(y) \in C(y)^*$ (i.e., $D(y) = \frac{1}{r(y)}$) is stratified if and only if $r(y) = \frac{d}{dy} s(y)$ for some $s(y) \in C(y)$.*

Proof. Suppose that $s(y)$ exists, then $D(s(y)) = D(y) \cdot \frac{d}{dy}s(y) = 1$. Apply now Corollary 1.3. On the other hand, if the equation is stratified then there is a $t \in C(y)$ with $D(t) = 1$. Then $1 = D(t) = D(y) \cdot \frac{d}{dy}t$ and $r(y) = \frac{d}{dy}t$. $\square$

We note that one can test whether $r(y)$ equals $\frac{d}{dy}s(y)$ by, for instance, considering the fractional expansion of $r(y)$. Moreover, if $s(y)$ exists, then infinitely many separable algebraic solutions are produced by the equation $s(y) = z + c$ with $c \in C$.

Suppose that the genus g of X is ≥ 1. Now D corresponds to an O_X-linear homomorphism $\Omega_X \to C(X)$, which is determined by its restriction $\ell : H^0(X, \Omega_X) \to C(X)$. We consider only those D's such that there exists $\omega \in H^0(X, \Omega_X)$ with $\ell(\omega) = 1$ and call such a D *special*.

Let x be a separating variable (i.e., $C(x) \subset C(X)$ is a finite separable extension) and write $H^0(X, \Omega_X) = V dx$, where $V \subset C(X)$ is a C-vector space of dimension g. Let $\ell(vdx) = 1$. Then $D = v^{-1}\frac{d}{dx}$.

According to 1.2, D is stratified if and only if $vdx = \sum_{i=0}^{p-2} v_i^p x^i dx$ for certain elements $v_i \in C(X)$. Equivalently $\mathcal{C}(vdx) = 0$, where $\mathcal{C} : H^0(X, \Omega_X) \to H^0(X, \Omega_X)$ is the Cartier operator.

We recall that $\mathcal{C}$ is defined by $\mathcal{C}(\sum_{i=0}^{p-1} v_i^p x^i dx) = v_{p-1}dx$. This definition does not depend on the choice of the separating variable x. Further $\mathcal{C}$ is the dual, by Serre duality, of the Frobenius operator acting upon $H^1(X, O_X)$.

Conclusion. The special vector fields D with $D^p = 0$ correspond to the lines $C\omega$ in $H^0(X, \Omega_X)$ such that $\mathcal{C}(\omega) = 0$.

We finish the autonomous case by some examples.

Proposition 2.3. *The autonomous Weierstrass differential equation is* $(x')^2 = x(x-1)(x-\lambda)$ *with* $\lambda \in C$, $\lambda \neq 0, 1$ *and* $p > 2$. *This equation is stratified if and only if the elliptic curve* $y^2 = x(x-1)(x-\lambda)$ *is supersingular.*

Proof. We note that the Weierstrass equation is (up to multiplication by an element in C^*) the only special equation for genus 1. From the above it follows that stratification is equivalent to the action of Frobenius on $H^1(X, O_X)$ is zero, or, equivalently, the elliptic curve is supersingular. We present another explicit proof.

The function field of the elliptic curve X is $F = C(y, x)$ and D is the standard holomorphic vector field $y\frac{d}{dx}$. Now $F = F^p(x)$ and $D(x) = y$. By 1.2 and 1.3, the equation is stratified if and only if the coefficient c_{p-1} of x^{p-1} in the polynomial $(x(x-1)(x-\lambda))^{(p-1)/2}$ is zero. A standard computation with Čech cohomology shows that the action of Frobenius Fr on the 1-dimensional C-vector space $H^1(X, O_X)$ is multiplication by c_{p-1}. This proves the statement. We note that $c_{p-1} = (-1)^{(p-1)/2} \cdot \sum_{i=0}^{(p-1)/2} \binom{(p-1)/2}{i}^2 \lambda^i$ is the well known Deuring polynomial for the supersingular elliptic curves. $\square$

Remark 2.4. For the situation in Proposition 2.3 we have in fact: $(y\frac{d}{dx})^p = \alpha(y\frac{d}{dx})$ with $\alpha = (-1)^{(p-1)/2} \cdot \sum_{i=0}^{(p-1)/2} \binom{(p-1)/2}{i}^2 \lambda^i$.

Example 2.5. *The Weierstrass equation* $(x')^2 = x^3 - x$ for $p = 3$.

The elliptic curve $y^2 = x^3 - x$ is supersingular and the equation admits a stratification. We want to make part (c) of Corollary 1.3 explicit and compute the solutions in $\overline{C(z)}^{sep}$.

Let $x \neq 0, 1, -1$ denote a solution in $\overline{C(z)}^{sep}$. Then $x' \neq 0$ and by differentiating $(x')^2 = x^3 - x$ one obtains $x'' = 1$. Thus $x' = z + k_1$ with $k_1 = h_1^3$ and $x = -z^2 + k_1 z + k_2$ with $k_2 = h_2^3$. Substitution in the equation and taking the third root leads to $h_2^3 - h_2 = z^2 - h_1^3 z + h_1^2$. A pair (h_1, h_2) satisfying this equation yields a solution $x = -h_1^2 + h_2$. The choice $h_1 = 0$ leads to $h_2^3 - h_2 = z^2$ which has a solution a in $\overline{C(z)}^{sep}$. Then a is a solution. After shifting $z \mapsto z + c$, $c \in C$ one finds other separable algebraic solutions. By taking $h_1 \in C$ one obtains a family of separable algebraic solutions.

We note that there is no rational solution $x \in C(z)$, since $C(x', x)$ is the function field of an elliptic curve.

3. Testing the existence of a stratification for some non-autonomous equations

Applying Theorem 1.1(c), stratification is equivalent to the existence of h with $D(h)=0$, $h \notin F^p$. This can be tested, since D is an explicit F^p-linear operator on F. In the genus zero case with $y' = D(y)$ in $K(y)=F$, it seems more efficient to compute $D^p(y)$ and apply Theorem 1.1(b).

3.1. Genus one, semi-autonomous, special D and p > 2

After replacing K by a finite separable extension, the function field F is $K(x,y)$ with $y^2 = x(x-1)(x-\lambda)$, $\lambda \in C$, $\lambda \neq 0, 1$ and $D = f(z)\cdot y\frac{d}{dx} + \frac{d}{dz}$ with $f(z) \in K^*$. We have to investigate whether $\ker(D,F) \neq F^p$ holds. For this, we may replace D by $E := y\frac{d}{dx} + f(z)^{-1}\frac{d}{dz}$.

The two parts $T_1 = y\frac{d}{dx}$ and $T_2 = \frac{1}{f(z)}\frac{d}{dz}$ of E commute and so $E^p = T_1^p + T_2^p$. Further T_1^p is a derivation on $C(x,y)$ without singularities and therefore equal to αT_1. In fact Remark 2.4 shows $\alpha = (-1)^{(p-1)/2}\cdot\sum_{i=0}^{(p-1)/2}\binom{(p-1)/2}{i}^2\lambda^i$. Furthermore T_2^p is a derivation on K and has the form $\beta(z)\frac{d}{dz}$ with $\beta(z) \in K$.

Suppose that the elliptic curve is supersingular (equivalently $\alpha = 0$).

Let $E(h)=0$. Then $E^p(h)=0$ and $\beta(z)\frac{d}{dz}(h)=0$.

If $\beta(z)\neq 0$, then $\frac{d}{dz}(h)=0$, $h \in K^p(x,y)$ and $T_1(h)=0$. The $K^p(x^p,y^p)$-linear operator T_1 on $K^p(x,y)$ is nilpotent and has only one Jordan block. Hence $h \in K^p(x^p,y^p)$ and *D is not stratified.*

If $\beta(z)=0$, then $T_2^p = 0$ and there is an element $h_2 \in K$ with $T_2(h_2)=1$. There is an element $h_1 \in C(x,y)$ with $T_1 h_1 = 1$. Then $E(h_1 - h_2)=0$ and *D is stratified.*

Moreover, the condition $\beta(z)=0$ (or $T_2^p = 0$) is equivalent to the existence of $h \in K$ with $\frac{1}{f(z)}\frac{d}{dx}h = 1$, which translates into: $f(z)$ is the derivative of some element in K. This can be tested.

Suppose that the elliptic curve is ordinary (equivalently $\alpha \neq 0$).

Let $E(h)=0$. Then $E^p(h)=0$. If $\beta(z) - \frac{\alpha}{f(z)} \neq 0$, then $y\frac{d}{dx}h = \frac{d}{dz}h = 0$ and $h \in K(x,y)^p$. *In this case D is not stratified.*

If $\beta(z) - \frac{\alpha}{f(z)} = 0$. Then T_1, as $C(x,y)^p$-linear operator on $C(x,y)$, satisfies $T_1^p - \alpha T_1 = 0$ and its eigenvalues are $\{n\gamma \mid 0 \le n < p\}$ where $\gamma^{p-1} = \alpha$. The operator T_2 satisfies the same equation and has the same eigenvalues. Take $h_1 \in C(x,y)^*$ with $T_1 h_1 = \gamma h_1$ and take $h_2 \in K^*$ with $T_2(h_2) = -\gamma h_2$. Then $E(h_1 h_2)=0$ and *D is stratified.*

Finally we have to test $\beta(z) - \frac{\alpha}{f(z)} = 0$ in the ordinary case.

This is equivalent to $T_2^p - \alpha T_2 = 0$. Since T_2 is a K^p-linear operator on K this is easy to test. □

3.2. Higher genus, semi-autonomous and special D

Here, a brief sketch is presented. After a finite separable extension of K we may suppose that the function field F is $K \cdot C(X)$, where $C(X)$ is the function field of a curve X over C and K a finite separable extension of $C(z)$. For a special derivation D one has to test whether $\ker(D,F) \neq F^p$. After replacing D by a suitable multiple E (as in the genus one case) one has $E = T_1 + T_2$ with T_1, T_2 commuting derivations, T_1 a special derivation on the field $C(X)$ with $T_1(z)=0$ and T_2 a derivation on K which is zero on $C(X)$. Combining the special autonomous equation corresponding to T_1 with properties of T_2 one can decide, as in the case of genus one, whether the original equation is stratified.

4. A Grothendieck–Katz conjecture for first order equations

Let the equation $f(y', y) = 0$ with absolute irreducible $f \in K[S,T]$ correspond to a curve X over a finite extension K of $\mathbb{C}(z)$ and a meromorphic derivation D. In the spirit of Katz (1982) we present the following conjecture.

Conjecture 4.1. *All solutions of $f(y', y) = 0$ are algebraic if and only if for almost all primes p the reduced equation f mod p is stratified.*

As in Katz (1982), reduction modulo p means that we replace $\mathbb{C}$ by a suitable finitely generated $\mathbb{Z}$-subalgebra R and divide R by a minimal prime ideal above pR.

First we will *sketch* a proof of the easy implication of Conjecture 4.1. In the terminology of Ngo et al. (2013) a solution of $f(y', y) = 0$ is a K-linear homomorphism $\phi : K[s, t, \frac{1}{d}] \to \mathcal{F}$, where $\mathcal{F} \supset K$ is an extension of differential fields, the field of constants of $\mathcal{F}$ is $\mathbb{C}$ and ϕ commutes with differentiation. The element $\phi(y) \in \mathcal{F}$ is the actual solution.

Suppose that all solutions of $f(y', y) = 0$ are algebraic. Then the kernel of D on the field of fractions F of $K[s, t, \frac{1}{d}]$ is strictly larger than $\mathbb{C}$. Indeed, otherwise the embedding $K[s, t, \frac{1}{d}] \subset F$ is a transcendental solution. Choose $u \in F$, $u \notin \mathbb{C}$ with $D(u) = 0$. Then u is transcendental over $\mathbb{C}(u)$ and F is a finite extension of the field $\mathbb{C}(u, z)$ and $D(u) = 0$.

It can be seen that, for almost all primes p, the reduction modulo p has the same feature. Namely: $\mathbb{C}$ is replaced by an algebraically closed field C of characteristic p; K is replaced by a finite separable extension $\tilde{K}$ of $C(z)$; the differential algebra $K[s, t, \frac{1}{d}]$ is replaced by $\tilde{K}[\bar{s}, \bar{t}, \frac{1}{\bar{d}}]$. The field of fractions of the latter is a finite separable extension of $C(\bar{u}, z)$ with $D(\bar{u}) = 0$, $D(z) = 1$. According to Theorem 1.1, part (c), this implies that f mod p is stratified.

In the proof of Proposition 4.2 below, the “easy implication” of the conjecture is given a more explicit proof.

Proposition 4.2. *Conjecture 4.1 holds for autonomous equations.*

Proof. Suppose that the equation has an algebraic solution. This is, according to Ngo et al. (2013), equivalent to the existence of a $t \in \mathbb{C}(X)$ with $D(t) = 1$. One may suppose that $t, t^{-1} \in R$ and for almost all primes p the reduction of t modulo p is nonzero. The reduced equation f mod p is stratified according to Corollary 1.3.

Suppose that f mod p is stratified for almost all primes p. Write $\mathbb{C}(X)$ as a finite extension of $\mathbb{C}(x)$. The derivation D has the form $D = a\frac{d}{dx}$. We have to investigate whether there exists $t \in \mathbb{C}(X)$ with $D(t) = 1$ or, equivalently, $dt = \omega := \frac{dx}{a}$.

According to Corollary 1.3, it is given that, for almost all primes p, the equation $D(t) = 1$ has a modulo p solution t_p. In other words, the differential form ω mod p is exact for almost all primes p. According to Y. André's work (André, 2004) (especially Proposition 6.2.1) and that of D.V. Chudnovsky & G.V. Chudnovsky (Chudnovsky and Chudnovsky, 1985), it follows that ω is exact. □

Remark. *Conjecture 4.1 holds for the Risch equation $y' = ay + b$ where $a, b \in K^*$ and K is a finite extension of $\mathbb{C}(z)$.*

Indeed, suppose that for almost all p, the reduced equation has infinitely many separable algebraic solutions. The difference of two of these is a nontrivial separable algebraic solution of the reduction modulo p of $y' = ay$. Hence $y' = ay$ has a nontrivial algebraic solution y_0. Replacing y by fy_0 yields the equation $f' = y_0^{-1}b$. The reduction modulo p of the latter has for almost all p a separable algebraic solution. It follows that $f' = y_0^{-1}b$ has an algebraic solution. Hence all solutions of $y' = ay + b$ are algebraic (compare André, 2004; Chudnovsky and Chudnovsky, 1985; van der Put, 2001).

Following a remark by a referee. Consider the equation (1) $y'' = ry$ with $r \in K$ and K a finite extension of $\mathbb{C}(z)$ and its Riccati equation (2) $u' + u^2 = r$.

Suppose that all solutions of (2) *are algebraic, then the same holds for* (1).

Indeed, let G denote the differential Galois group of (1). A line $\mathbb{C}y$ in the solution space V of (1) corresponds to a solution $u = \frac{y'}{y}$ of (2). Since u is algebraic, the line $\mathbb{C}y$ is invariant under a closed subgroup of finite index of G. In particular, every line of V is invariant under G^o, the component of the identity of G. Since $G \subset \mathrm{SL}(2, \mathbb{C})$ one has $G^o = \{1\}$ and G is finite. Thus all solutions of (1) are algebraic (compare van der Put and Singer, 2003, Lemma 4.8 for more details). □

Let $(1)_p$ and $(2)_p$ denote the reductions of (1) and (2) module a prime p. Suppose that $(1)_p$ has p-curvature zero for almost all p. Then $(2)_p$ has infinitely many separable algebraic solutions.

Assume that Conjecture 4.1 holds for equation (2).

Then all solutions of (2) are algebraic and the same holds for (1). Thus Conjecture 4.1 for (2) implies the usual Grothendieck–Katz conjecture for (1).

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